MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes VII

Ng Hoi Dong

L'Hopital's Rule

Suppose $b > a$ and $c \in (a, b)$, $f, g : (a, b) \to \mathbb{R}$ is a function differentiable on $(a, b) \setminus \{c\}$.

Suppose $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ OR $\lim_{x\to c} f(x) = \pm\infty = \lim_{x\to c} g(x)$.

If $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ EXISTS, then we will have

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

In fact, the Rule is still true when c is replaced by c^+, c^- or $\pm\infty$.

Taylor's Polynomial

Let f be a function which is k-times differentiable on some interval I, and let $a \in I$.

The k-th Taylor's Polynomial of f centered at a is defined by

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Useful Taylor's Polynomial

Let $P_k(x)$ be the k-th Taylor's Polynomial of f centered at 0.

$f(x)$	Taylor's Polynomial
$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$	$P_k(x) = \begin{cases} a_0 + a_1x + a_2x^2 + \ldots + a_kx^k & ,k < n \\ a_0 + a_1x + a_2x^2 + \ldots + a_nx^n & ,k \geq n \end{cases}$
e^x	$P_k(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!}$
$\cos x$	$P_{2k}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{2k!}$
$\sin x$	$P_{2k+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{2k+1!}$
$\ln(1-x)$	$P_k(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^k}{k}$
$\frac{1}{1-x}$	$P_k(x) = 1 + x + x^2 + \cdots + x^k$

Question

(Q1) Use L'Hopital's Rule to evaluate the following limits.

- **(a)** $\lim\limits_{x \to 0} \dfrac{\sin 3x}{\sin 5x}$.
- **(b)** $\lim\limits_{x \to 0} \dfrac{\ln \cos 2x}{\ln \cos x}$.
- **(c)** $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right)$.
- **(d)** $\lim\limits_{x \to 0^+} x^{\frac{1}{1+\ln x}}$.
- **(e)** $\lim\limits_{x \to +\infty} x \left(\dfrac{\pi}{2} - \tan^{-1} x \right)$.
- **(f)** $\lim\limits_{x \to +\infty} \left(e^x + x \right)^{\frac{1}{x}}$.

(Q2) Find

- **(a)** 8-th Taylor's Polynomial of $f(x) = e^{x^2}$ centered at 0.
- **(b)** 5-th Taylor's Polynomial of $f(x) = xe^x$ centered at 0.
- **(c)** 10-th Taylor's Polynomial of $f(x) = \cos x$ centered at π.
- **(d)** 6-th Taylor's Polynomial of $f(x) = \ln(1 + x)$ centered at 0.
- **(e)** 5-th Taylor's Polynomial of $f(x) = \dfrac{1}{2 + x}$ centered at 0.
- **(f)** 6-th Taylor's Polynomial of $f(x) = \sinh x \overset{\text{def}}{=} \dfrac{e^x - e^{-x}}{2}$ centered at 0.
- **(g)** 7-th Taylor's Polynomial of $f(x) = \tan^{-1} x$ centered at 0.
- **(h)** 3-th Taylor's Polynomial of $f(x) = \sqrt{1 + x}$ centered at 0.
- **(i)** 6-th Taylor's Polynomial of $f(x) = \sqrt{1 + x^2}$ centered at 0.

(Q3) Find all horizontal, vertical and oblique asymptotes of $y = \dfrac{x|x| + x^2 + x + 1}{|x| - 2}$.

(Q4) Sketch the graph of $y = \left(1 + 3x\right) e^{-2x}$.

Answer

(A1) (Please remember to check the condition!)

(a)

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$$

as $x \to 0$,

$\sin 3x \to 0$ and $\sin 5x \to 0$.

$$\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} \lim_{x\to 0}\frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$$

(b)

$$\lim_{x\to 0}\frac{\ln\cos 2x}{\ln\cos x}$$

as $x \to 0$,

$\ln\cos 2x \to -\infty$ and $\ln\cos x \to -\infty$.

$$\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} \lim_{x\to 0}\frac{-\frac{2\sin 2x}{\cos 2x}}{-\frac{\sin x}{\cos x}}$$

$$= \quad 2\lim_{x\to 0}\frac{\tan 2x}{\tan x}$$

as $x \to 0$

$\tan 2x \to 0$ and $\tan x \to 0$.

$$\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} 2\lim_{x\to 0}\frac{2\sec 2x}{\sec x} = 4$$

(c)

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$

$$= \quad \lim_{x\to 0}\frac{e^x-x-1}{x\left(e^x-1\right)}$$

as $x \to 0$,

$e^x-x-1 \to 0$ and $x\left(e^x-1\right) \to 0$

$$\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} \lim_{x\to 0}\frac{e^x-1}{xe^x+e^x-1}$$

as $x \to 0$,

$e^x-1 \to 0$ and $xe^x+e^x-1 \to 0$

$$\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} \lim_{x\to 0}\frac{e^x}{xe^x+2e^x} = \frac{1}{2}$$

(d) Note that $x^{\frac{1}{1+\ln x}} = e^{\ln\left(x^{\frac{1}{1+\ln x}}\right)} = e^{\frac{\ln x}{1+\ln x}}$ and

$$\lim_{x\to 0}\frac{\ln x}{1+\ln x}$$

as $x \to 0$,

$\ln x \to -\infty$ and $1+\ln x \to -\infty$.

$$\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} \lim_{x\to 0}\frac{\frac{1}{x}}{\frac{1}{x}} = 1$$

and hence, $\lim_{x\to 0} x^{\frac{1}{1+\ln x}} = e^1 = e$.

(e)

$$\begin{aligned}
& \lim_{x\to\infty} x\left(\frac{\pi}{2} - \tan^{-1} x\right) \\
= & \lim_{x\to\infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{\frac{1}{x}} \qquad \text{as } x \to 0, \\
& \qquad \frac{\pi}{2} - \tan^{-1} x \to 0 \text{ and } \frac{1}{x} \to 0. \\
\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} & \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\
= & \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1
\end{aligned}$$

(f) Note that $(e^x + x)^{\frac{1}{x}} = e^{\ln\left[(e^x+x)^{\frac{1}{x}}\right]} = e^{\frac{\ln(e^x+x)}{x}}$ and

$$\begin{aligned}
& \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \qquad \text{as } x \to \infty, \\
& \qquad \ln\left(e^x + x\right) \to \infty \text{ and } x \to \infty \\
\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} & \lim_{x\to\infty} \frac{\frac{e^x+1}{e^x+x}}{1} = 1
\end{aligned}$$

and so $\lim_{x\to\infty} \left(e^x + x\right)^{\frac{1}{x}} = e$.

(A2) (It will be better if you can show the formula in the first page)

For some question, it is difficult to find the polynomial directly from definition.

But you CAN find it by using definition at least for (c),(d),(e),(f),(h).

(a) Note that 4-th Taylor's Polynomial of $g(w) = e^w$ centered at 0 is

$$1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \frac{w^4}{24}.$$

Substitute $w = x^2$ we will have 8-th Taylor's Polynomial of $f(x) = e^{x^2}$ centered at 0 is

$$1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}.$$

(b) Note that 4-th Taylor's Polynomial of $g(w) = e^w$ centered at 0 is

$$1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \frac{w^4}{24}.$$

Multiply w on g, we will have 8-th Taylor's Polynomial of $f(x) = xe^x$ centered at 0 is

$$x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24}.$$

(c) There are two method:

(Method 1) Note that 10-th Taylor's Polynomial of $g(w) = \cos w$ centered at π is

$$1 - \frac{w^2}{2} + \frac{w^4}{4!} - \frac{w^6}{6!} + \frac{w^8}{8!} - \frac{w^{10}}{10!}$$

Note that $-\cos(x - \pi) = \cos x$, substitute $w = x - \pi$,
we will have 10-th Taylor's Polynomial of $f(x) = \cos x$ centered at π is

$$-1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!} + \frac{(x-\pi)^{10}}{10!}$$

(Method 2) Note that

$$\begin{aligned}
f(x) &= \cos x, & f(\pi) &= -1,\\
f'(x) &= -\sin x, & f'(\pi) &= 0,\\
f''(x) &= -\cos x, & f''(\pi) &= 1,\\
f'''(x) &= \sin x, & f'''(\pi) &= 0,\\
f^{(4)}(x) &= \cos x, & f^{(4)}(\pi) &= -1,\\
f^{(5)}(x) &= -\sin x, & f^{(5)}(\pi) &= 0,\\
f^{(6)}(x) &= -\cos x, & f^{(6)}(\pi) &= 1,\\
f^{(7)}(x) &= \sin x, & f^{(7)}(\pi) &= 0,\\
f^{(8)}(x) &= \cos x, & f^{(8)}(\pi) &= -1,\\
f^{(9)}(x) &= -\sin x, & f^{(9)}(\pi) &= 0,\\
f^{(10)}(x) &= -\cos x, & f^{(10)}(\pi) &= 1.
\end{aligned}$$

We will have 10-th Taylor's Polynomial of $f(x) = \cos x$ centered at π is

$$-1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!} + \frac{(x-\pi)^{10}}{10!}.$$

(d) There are two method:

(Method 1) Note that 6-th Taylor's Polynomial of $g(w) = \ln(1-w)$ centered at 0 is

$$w + \frac{w^2}{2} + \frac{w^3}{3} + \frac{w^4}{4} + \frac{w^5}{5} + \frac{w^6}{6}.$$

Substitute $w = -x$, we will have 6-th Taylor's Polynomial of $f(x) = \ln(1-x)$ centered at 0 is

$$-x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^6}{6}.$$

(Method 2) Note that

$$\begin{aligned}
f(x) &= \ln(1+x), & f(0) &= 0,\\
f'(x) &= \frac{1}{1+x}, & f'(0) &= 1,\\
f''(x) &= \frac{-1}{(1+x)^2}, & f''(0) &= -1,\\
f'''(x) &= \frac{2}{(1+x)^3}, & f'''(0) &= 2,\\
f^{(4)}(x) &= \frac{-3!}{(1+x)^4}, & f^{(4)}(0) &= -3!,\\
f^{(5)}(x) &= \frac{4!}{(1+x)^5}, & f^{(5)}(0) &= 4!,\\
f^{(6)}(x) &= \frac{-5!}{(1+x)^6}, & f^{(6)}(0) &= -5!.
\end{aligned}$$

We will have 6-th Taylor's Polynomial of $f(x) = \ln(1-x)$ centered at 0 is

$$-x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^6}{6}.$$

(e) There are two method:

(Method 1) Note that 5-th Taylor's Polynomial of $g(w) = \dfrac{1}{1-w}$ centered at 0 is

$$1 + w + w^2 + w^3 + w^4 + w^5.$$

Note that $\dfrac{1}{x+2} = \dfrac{1}{2}\dfrac{1}{1+\frac{x}{2}}$. Substitute $w = -\dfrac{x}{2}$ and multiply $\dfrac{1}{2}$ on g, We will have 5-th Taylor's Polynomial of $f(x) = \dfrac{1}{x+2}$ centered at 0 is

$$\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \frac{x^5}{64}.$$

(Method 2) Note that

$$\begin{aligned}
f(x) &= \frac{1}{x+2}, & f(0) &= \frac{1}{2},\\
f'(x) &= \frac{-1}{(x+2)^2}, & f'(0) &= -\frac{1}{4},\\
f''(x) &= \frac{2}{(x+2)^3}, & f''(0), &= \frac{2}{8},\\
f'''(x) &= \frac{-3!}{(x+2)^4} & , f'''(0) &= -\frac{3!}{16},\\
f^{(4)}(x) &= \frac{4!}{(x+2)^5} & , f^{(4)}(0) &= \frac{4!}{32},\\
f^{(5)}(x) &= \frac{-5!}{(x+2)^6} & , f^{(5)}(0) &= \frac{-5!}{64}.
\end{aligned}$$

We will have 5-th Taylor's Polynomial of $f(x) = \dfrac{1}{x+2}$ centered at 0 is

$$\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \frac{x^5}{64}.$$

(f) There are two method:

(Method 1) Note that 6-th Taylor's Polynomial of $g(w) = e^w$ centered at 0 is

$$1 + w + \frac{w^2}{2} + \frac{w^3}{3!} + \frac{w^4}{4!} + \frac{w^5}{5!} + \frac{w^6}{6!}.$$

Substitute $y = -w$ the 6-th Taylor's Polynomial of $h(y) = e^{-y}$ centered at 0 is

$$1 - y + \frac{y^2}{2} - \frac{y^3}{3!} + \frac{y^4}{4!} - \frac{y^5}{5!} + \frac{y^6}{6!}.$$

Note that $f(x) = \dfrac{1}{2}\big(g(x) - h(x)\big)$

We will have 6-th Taylor's Polynomial of $f(x) = \sinh x$ centered at 0 is

$$x + \frac{x^3}{3!} + \frac{x^5}{5!}.$$

(Method 2) Note that

$$\begin{aligned}
f(x) &= \frac{e^x - e^{-x}}{2}, & f(0) &= 0,\\
f'(x) &= \frac{e^x + e^{-x}}{2}, & f'(0) &= 1,\\
f''(x) &= \frac{e^x - e^{-x}}{2}, & f''(0) &= 0,\\
f'''(x) &= \frac{e^x + e^{-x}}{2}, & f'''(0) &= 1,\\
f^{(4)}(x) &= \frac{e^x - e^{-x}}{2}, & f^{(4)}(0) &= 0,\\
f^{(5)}(x) &= \frac{e^x + e^{-x}}{2}, & f^{(5)}(0) &= 1,\\
f^{(6)}(x) &= \frac{e^x - e^{-x}}{2}, & f^{(6)}(0) &= 0.
\end{aligned}$$

We will have 6-th Taylor's Polynomial of $f(x) = \sinh x$ centered at 0 is

$$x + \frac{x^3}{3!} + \frac{x^5}{5!}.$$

(g) If $f(x) = \tan^{-1} x$, then $f'(x) = \dfrac{1}{1+x^2}$ and $f(0) = 0$.

Note that 3-th Taylor's Polynomial of $g(w) = \dfrac{1}{1-w}$ centered at 0 is

$$1 + w + w^2 + w^3.$$

Substitute $w = -x^2$,

we have 6-th Taylor's Polynomial of $h(x) = f'(x) = \dfrac{1}{1+x^2}$ centered at 0 is

$$1 - x^2 + x^4 - x^6.$$

Then we know

$$\begin{aligned} f'(0) &= h(0) = 1, & f''(0) &= h'(0) = 0, & f'''(0) &= h''(0) = -2, \\ f^{(4)}(0) &= h'''(0) = 0, & f^{(5)}(0) &= h^{(4)}(0) = 4!, & f^{(6)}(0) &= h^{(5)}(0) = 0, \\ & & f^{(7)}(0) &= h^{(6)}(0) = -6!. & & \end{aligned}$$

We will have 7-th Taylor's Polynomial of $f(x) = \tan^{-1} x$ centered at 0 is

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}.$$

(h) Note that

$$\begin{aligned} f(x) &= (1+x)^{\frac{1}{2}}, & f(0) &= 1, \\ f'(x) &= \frac{1}{2}(1+x)^{-\frac{1}{2}}, & f'(0) &= \frac{1}{2}, \\ f''(x) &= \frac{-1}{4}(1+x)^{-\frac{3}{2}}, & f''(0) &= \frac{-1}{4}, \\ f'''(x) &= \frac{3}{8}(1+x)^{-\frac{5}{2}}, & f'''(0) &= \frac{3}{8}. \end{aligned}$$

We will have 3-th Taylor's Polynomial of $f(x) = \sqrt{1+x}$ centered at 0 is

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

(i) Using (h), we will have 6-th Taylor's Polynomial of $f(x) = \sqrt{1+x^2}$ centered at 0 is

$$1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6.$$

(A3) Note that

$$y = f(x) = \frac{x|x| + x^2 + x + 1}{|x| - 2} = \begin{cases} \dfrac{2x^2 + x + 1}{x - 2}, & \text{if } x \geq 0 \\ -\dfrac{x+1}{x+2}, & \text{if } x < 0 \end{cases}.$$

Note that the denominator of the expression of y will be 0 if $x = \pm 2$.

That is, $f(x) \to \pm\infty$ as $x \to 2$ or $x \to -2$.

Therefore, $x = 2$ and $x = -2$ is a vertical asymptotes of $y = f(x)$.

Moreover, $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} -\dfrac{x+1}{x+2} = -1$.

Hence, $y = -1$ is a horizontal asymptotes of $y = f(x)$.

Using long division, we know $\dfrac{2x^2 + x + 1}{x + 2} = (2x + 5) - \dfrac{11}{x + 2}$.

Therefore, $\lim\limits_{x\to\infty}\left[f(x)-(2x+5)\right]=\lim\limits_{x\to\infty}-\dfrac{11}{x+2}=0.$

Hence, $y=2x+5$ is a oblique asymptotes of $y=f(x)$.

(A4) *(Need to find y', y'', turning and inflection point and asymptotes.)*

(Write down ALL the details please.)

(Step 1) Note that

$$\begin{aligned}
y &= \left(1+3x\right)e^{-2x} \\
y' &= -2\left(1+3x\right)e^{-2x}+3e^{-2x} \\
&= \left(1-6x\right)e^{-2x} \\
y'' &= -2\left(1-6x\right)e^{-2x}-6e^{-2x} \\
&= \left(-8+12x\right)e^{-2x}
\end{aligned}$$

(Step 2) When $x=0$, $y=1$.

By solving $y=0$, we know the root of y are $x=-\dfrac{1}{3}$.

By solving $y'=0$, we know the turning point is $\left(\dfrac{1}{6},\dfrac{3}{2}e^{-\frac{1}{3}}\right)$.

By solving $y''=0$, we know the infection point is $\left(\dfrac{2}{3},3e^{-\frac{4}{3}}\right)$.

x	$\left(-\infty,-\frac{1}{3}\right)$	$-\frac{1}{3}$	$\left(-\frac{1}{3},\frac{1}{6}\right)$	$\frac{1}{6}$	$\left(\frac{1}{6},\frac{2}{3}\right)$	$\frac{2}{3}$	$\left(\frac{3}{2},+\infty\right)$
y	-ve	0	+ve	$\frac{3}{2}e^{-\frac{1}{3}}$	+ve	$3e^{-\frac{4}{3}}$	+ve
y'	+ve	+ve	+ve	0 (Local max)	-ve	-ve	-ve
y''	-ve	-ve	-ve	-ve	-ve	0	+ve

(Step 3) The ONLY (why?) asymptotes of y is $y=0$ as $x\to\infty$.

(Step 4) Sketch the graph.